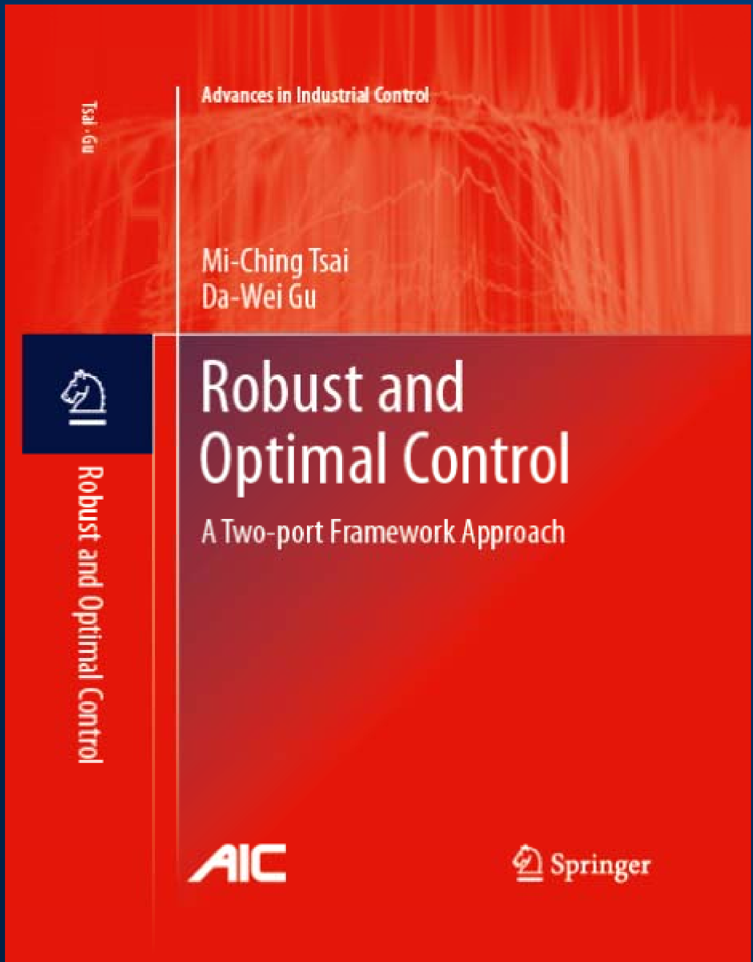

Robust and Optimal Control

A Two-port Framework Approach

Coprime Factorization

Coprimeness for the real numbers

One can start with the simplest case of real numbers. Given a real rational number $r = \frac{n}{d}$, where d and n are two integers. If the greatest common divisor (g.c.d.) of the pair of integers (d, n) is 1, then d and n are called coprime and the $r = \frac{n}{d}$ called the coprime factorization of r over the integers. It is well known that if a pair of integers (d, n) are coprime, there exists a row vector of two integers $[\tilde{x} \quad \tilde{y}]$ such that $\tilde{x}d - \tilde{y}n = 1$.

Example :

For $r = \frac{3}{2}$, it can be found that $\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1$

$[\tilde{x} \quad -\tilde{y}]$ is called the left inverse of $\begin{bmatrix} d \\ n \end{bmatrix}$

⇒ $1.5 = \frac{3}{2}$ is the coprime factorization of the real rational number 1.5

Coprimeness over a ring of polynomials with real coefficients

Proceeding forward, two polynomials are called coprime if they do not share common zeros.

Example : Check if $F(s)=\dfrac{n(s)}{d(s)}=\dfrac{s+2}{s+1}$ is coprime.

Trivially, this is a coprime factorization over the polynomial ring since one can find that $\begin{bmatrix} s+1 & -s \end{bmatrix}\begin{bmatrix} s+1 \\ s+2 \end{bmatrix}=1.$

Example : Check if $F(s)=\dfrac{n(s)}{d(s)}=\dfrac{s^2+5s+6}{s^2+4s+3}$ is coprime.

The pair of $(n(s),d(s))$ is not coprime since s=-3 is a common zero. It can also be seen that this factorization is *reducible* as that

$$F(s)=\frac{s^2+5s+6}{s^2+4s+3}=\frac{(s+2)\,\cancel{(s+3)}}{(s+1)\,\cancel{(s+3)}}$$

$$\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix}\begin{bmatrix} (s+2)(s+3) \\ (s+1)(s+3) \end{bmatrix}=1$$

$$if\ \ s=-3$$

$$\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\neq 1$$

Coprime factorization of a rational function

Two stable rational functions *M(s)* and *N(s)* are coprime if there exist two stable rational functions $\left(\tilde{X}(s), \tilde{Y}(s)\right)$ such that $\begin{bmatrix} \tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \begin{bmatrix} M(s) \\ -N(s) \end{bmatrix} = I$

Example : $T(s) = \dfrac{s-2}{s-3} = \dfrac{N(s)}{M(s)}$ *where* $M(s) = \dfrac{s-3}{s+1}, N(s) = \dfrac{s-2}{s+1} \in RH_\infty$

One has $\begin{bmatrix} \dfrac{-4s-1}{s+1} & -\dfrac{5s+1}{s+1} \end{bmatrix} \begin{bmatrix} \dfrac{s-3}{s+1} \\ -\dfrac{s-2}{s+1} \end{bmatrix} = 1$ coprime

Till now, one has discussed the coprimeness of SISO systems over integers, polynomials, and stable rational functions, respectively. In the following, one will expand the coprimeness over stable rational function matrices which will be introduced for general MIMO cases in the development of control system analysis and synthesis.

Coprime factorization over RH_∞

Given a transfer function matrix *T(s)*, a basic problem is to find four transfer function matrices *N(s)*, *M(s)*, $\tilde{M}(s)$, and $\tilde{N}(s)$ in RH_∞ such that

$$T(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

where the pair of $\{M(s), N(s)\}$ is right coprime, and $\{\tilde{M}(s), \tilde{N}(s)\}$ left coprime. Such coprime factors do exist.

Example :

Find the coprime factors of $T(s) = \begin{bmatrix} \dfrac{s+2}{(s+3)^2} & 0 \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix}$