Chapter 9

Problem 1

Consider a unity feedback system with a proportional gain controller K, where $K=3$ and the plant under control $G(s)=\frac{1}{s-2}$. Compute a normalized coprime factorization of $G(s)$. Considering perturbations Δ_N and Δ_M of the normalized coprime factors of $G(s)$, compute the stability radius ε with regard to the perturbations on coprime factors.

Let $N(s)=\frac{1}{s+\lambda}$ and $M(s)=\frac{s-2}{s+\lambda}$.

We can get $\tilde{N}(s)=\frac{1}{-s+\lambda}$ and $\tilde{M}(s)=\frac{-s-2}{-s+\lambda}$.

Normalized coprime factorization

$$\tilde{N}N+\tilde{M}M=1 \quad \Rightarrow \quad \frac{1}{(s+\lambda)(-s+\lambda)}+\frac{(s-2)(-s-2)}{(s+\lambda)(-s+\lambda)}=1$$

$$\Rightarrow \quad \frac{5-s^2}{\lambda^2-s^2}=1 \quad \Rightarrow \quad \lambda=\pm\sqrt{5}$$

For $N,M\in RH_\infty,\quad \lambda=\sqrt{5}.$

We obtain the normalized coprime factorizations of *G* as

$$N(s)=\frac{1}{s+\sqrt{5}} \text{ and } M(s)=\frac{s-2}{s+\sqrt{5}}$$

From the robust stability condition of right coprime factorization,

$$\left\|M^{-1}\left(I+KP_0\right)^{-1}\begin{bmatrix}K & I\end{bmatrix}\right\|_\infty = \left\|\frac{s+\sqrt{5}}{s-2}\left(I+\frac{3}{s-2}\right)^{-1}\begin{bmatrix}3 & 1\end{bmatrix}\right\|_\infty$$

$$= \left\|\frac{s+\sqrt{5}}{s+1}\begin{bmatrix}3 & 1\end{bmatrix}\right\|_\infty = 3\sqrt{5}$$

Let $\varepsilon = 3\sqrt{5}$.

For $\left\|M^{-1}\left(I+KP_0\right)^{-1}\begin{bmatrix}K & I\end{bmatrix}\right\|_\infty \le 3\sqrt{5}$,

the system is robust stabilized as $\|\Delta\|_\infty \le \dfrac{1}{3\sqrt{5}}$.

Problem 2

For a given SCC plant $P(s) \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|c:c} 1 & 0 & 1 & 1 \\ 2 & -3 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hdashline 1 & 2 & 3 & 0 \end{array}\right]$ and $\gamma = 5$, compute a sub-optimal H-infinity controller using the CSD approach.

Step 1.

$$P_\gamma(s) \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline \frac{C_1}{\gamma} & \frac{D_{11}}{\gamma} & \frac{D_{12}}{\gamma} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & 0 \end{array}\right] = \left[\begin{array}{cc|c:c} 1 & 0 & 1 & 1 \\ 2 & -3 & 0 & 0 \\ \hline \frac{2}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ \hdashline 1 & 2 & 3 & 0 \end{array}\right]$$

To construct H_∞ controllers following the procedure of a right CSD associated with a left CSD, by (9.11), the particular right coprime factorization is given

$$G_1(s) \stackrel{s}{=} \left[\begin{array}{c|cc} A_p + B_{p2}F_{u1} + B_{p1}F_w & B_{p2}W_{uu} + B_{p1}W_{wu} & B_{p1}W_{ww} \\ \hline C_{p1} + D_{p12}F_{u1} + D_{p11}F_w & D_{p12}W_{uu} + D_{p11}W_{wu} & D_{p11}W_{ww} \\ F_w & W_{wu} & W_{ww} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} -1.67 & -0.48 & 5 & 1 \\ 2 & -3 & 0 & 0 \\ \hline 0.4 & 0 & 0 & 0 \\ -0.56 & 0.11 & 1 & 0 \\ 0.11 & -0.02 & 0 & 1 \end{array}\right],$$

$$\tilde{G}_2(s) \stackrel{s}{=} \left[\begin{array}{c|cc} A_p + B_{p2}F_{u1} + B_{p1}F_w & B_{p2}W_{uu} + B_{p1}W_{wu} & B_{p1}W_{ww} \\ \hline F_{u1} & W_{uu} & 0 \\ C_{p2} + D_{p21}F_w & D_{p21}W_{wu} & D_{p21}W_{ww} \end{array}\right] = \left[\begin{array}{cc|cc} -1.67 & -0.48 & 5 & 1 \\ 2 & -3 & 0 & 0 \\ \hline -2.78 & -0.46 & 5 & 0 \\ 1.33 & 1.94 & 0 & 3 \end{array}\right],$$

$$M_\infty(s) \stackrel{s}{=} \left[\begin{array}{c|cc} A_p + B_{p2}F_{u1} + B_{p1}F_w & B_{p2}W_{uu} + B_{p1}W_{wu} & B_{p1}W_{ww} \\ \hline F_{u1} & W_{uu} & 0 \\ F_w & W_{wu} & W_{ww} \end{array}\right] = \left[\begin{array}{cc|cc} -1.67 & -0.48 & 5 & 1 \\ 2 & -3 & 0 & 0 \\ \hline -2.78 & -0.46 & 5 & 0 \\ 0.11 & -0.02 & 0 & 1 \end{array}\right],$$

where $F_I = \begin{bmatrix} F_{u1} \\ F_w \end{bmatrix}$ and $W_I = \begin{bmatrix} W_{uu} & 0 \\ W_{wu} & W_{ww} \end{bmatrix}$ are found such that $G_1 \in RH_\infty$ is J-lossless.

From (9.15)-(9.19), it can be verified that G_1 is J-lossless if

$$W_I = \begin{bmatrix} W_{uu} & 0 \\ W_{wu} & W_{ww} \end{bmatrix}$$

$$= W_I = \begin{bmatrix} \left[D_{p12}^T\left(I - D_{p11}D_{p11}^T\right)^{-1} D_{p12}\right]^{-\frac{1}{2}} & 0 \\ \left(I - D_{p11}^T D_{p11}\right)^{-1} D_{p11}^T D_{p12}\left[D_{p12}^T\left(I - D_{p11}D_{p11}^T\right)^{-1} D_{p12}\right]^{-\frac{1}{2}} & \left(I - D_{p11}^T D_{p11}\right)^{-\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$X = \mathrm{Ric}(H_X) \geq 0,$