Problem 1

Prove that all the eigenvalues $\lambda(H)$ of a Hamiltonian matrix H are symmetric to the $j\omega$-axis.

For Hamiltonian matrix H, we know

$$\begin{cases} JH = -H^T J \\ J^T = -J \\ J^{-1} = -J \end{cases} \text{, where } J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

From $JH = -H^T J$, we know

$$JHJ^{-1} = -JHJ = J^{-1}HJ = -H^T \Rightarrow J^{-1}HJ = -H^T.$$

By simililarity transformation, we know $\lambda(H) = \lambda(-H^T)$; hence, the eigenvalues $\lambda(H)$ of a Hamiltonian matrix H are symmetric to the $j\omega$-axis.

Problem 2

Determine the rank of $A = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 1 & 9 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 1 & 9 \end{bmatrix}$$

$$\xrightarrow{\text{R3=R1-R3}} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 4 & -1 & 2 \\ 0 & 0 & 4 & -8 \end{bmatrix}$$

$$\xrightarrow[\text{R2=2*R1-R2}]{\text{R3=R3/4}} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[\text{R3=R2-R3}]{\text{R2=R2/11}} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\therefore$ The rank of A is 3.

Problem 3.

Let $Q=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$, $R=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $b=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, Utilize the least square approach to solve $Ax=b$ where $A=QR$.

Concept: least square method, QR factorization

Step 1.

Multiplying by A^T on both sides of the equation $Ax=b$,

$$A^T Ax = A^T b$$
$$\Rightarrow R^T Q^T QRx = R^T Q^T b \quad (A = QR).$$

For the orthogonal matrix Q ($Q^T Q = I$) and above equation, one can obtain

$$R^T Rx = R^T Q^T b$$
$$\Rightarrow Rx = Q^T b.$$

Step 2.

Because the matrix R is nonsingular,

$$x = R^{-1} Q^T b$$
$$\Rightarrow x = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}.$$

Note: If *A* is $m \times n$ and left-invertible,

- R is $n \times n$ and upper triangular with $r_{ii} > 0$.
- Q is $m \times n$ and orthogonal $Q^T Q = I$.

Problem 4.

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

Find the response of $x_1(t)$ and $x_2(t)$

Concept : state response of LTI system

Step 1.

According to Equation (2.53) on Page 18, the state response of a LTI system in the time domain is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \,.$$

Step 2.

Because the input $u(t)$ is zero, the state response can be

$$x(t) = e^{At}x_0 = \begin{bmatrix} e^{-t} & 0 \\ \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} & e^{-4t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} e^{-t}x_{10} \\ (\frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t})x_{10} + e^{-4t}x_{20} \end{bmatrix}.$$