Robust and Optimal Control

A Two-port Framework Approach

Bounded Real Lemma

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Bounded Real Lemma

$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RH_\infty \qquad G^\sim(s) \overset{s}{=} \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \qquad G_\gamma = \frac{1}{\gamma} G$$

$$I - G_\gamma^\sim G_\gamma \overset{s}{=} \left[\begin{array}{cc|c} A & BB^T & -B\dfrac{D^T}{\gamma} \\ 0 & -A^T & \dfrac{C^T}{\gamma} \\ \hline \dfrac{C}{\gamma} & \dfrac{DB^T}{\gamma} & I - \dfrac{DD^T}{\gamma} \end{array}\right] = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

$$(I - G_\gamma^\sim G_\gamma)^{-1} \overset{s}{=} \left[\begin{array}{c|c} \hat{A} - \hat{B}\hat{D}^{-1}\hat{C} & \hat{B}\hat{D}^{-1} \\ \hline -\hat{D}^{-1}\hat{C} & \hat{D}^{-1} \end{array}\right]$$

Bounded Real Lemma

$$\hat{A}-\hat{B}\hat{D}^{-1}\hat{C}=\begin{bmatrix} A & BB^T \\ 0 & -A^T \end{bmatrix}-\begin{bmatrix} -B\dfrac{D^T}{\gamma} \\ \dfrac{C^T}{\gamma} \end{bmatrix}\begin{bmatrix} I-\dfrac{DD^T}{\gamma} \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{C}{\gamma} & \dfrac{DB^T}{\gamma} \end{bmatrix}$$

$$=\begin{bmatrix} A+B(\gamma^2 I-D^T D)^{-1}D^T C & \gamma B(\gamma^2 I-D^T D)^{-1}\gamma B^T \\ -\dfrac{C^T}{\gamma}(I+D(\gamma^2 I-D^T D)^{-1}D^T)\dfrac{C}{\gamma} & -(A+B(\gamma^2 I-D^T D)^{-1}D^T C)^T \end{bmatrix}$$

$$H_\gamma=\begin{bmatrix} A+BR^{-1}D^T C & \gamma BR^{-1}\gamma B^T \\ -\dfrac{C^T}{\gamma}(I+DR^{-1}D^T)\dfrac{C}{\gamma} & -\left(A+BR^{-1}D^T C\right)^T \end{bmatrix}$$

$$R=\gamma^2 I-D^T D$$

Thus the poles of $(I-G_\gamma^{\sim}(s)G_\gamma(s))^{-1}$ **are contained in the eigenvalues of** H_γ **.**